An early history of difficult multiplication and division

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Introduction

Multiplication and division have in general been much more difficult to perform than addition and subtraction. Perhaps, if we could find some device for reducing multiplication and division to addition and subtraction, computational loads could be lightened. One such device is that of logarithms of course. This note outlines another such device with a brief outline of its history; that being the use of trigonometric tables to effect multiplication and division. The technique is called *prosthaphaeresis* a Greek word meaning addition and subtraction.

We would not use this technique today because time has moved on and we have better, more advanced ways of calculating. However, in its time, it was a very practical way of doing difficult multiplication and division. It could receive mention in the upper secondary school when trigonometric identities are discussed to show their practical use in times past.

Background

The first trigonometric tables, although not called such, were chord and half-chord tables (see Figure 1). The first such tables were based on work by Aristarchus (about 320–250 BC) a Greek astronomer who was born in Samos, and the work of Apollonius (about 261–190 BC), a Greek mathematician who was born in Perga on the southern coast of what is now Turkey. Hipparchus (about 190–120 BC) of Nicaea now Iznik, in what is now northwest Asian Turkey compiled the first of such chord tables (not now extant). He is credited with being the greatest of the Greek astronomers and the chord table was only one among other achievements. Over three centuries later, Ptolemy (born about AD 75), in his book *Almagest*, compiled a table of chords — essentially a chord table as in upper Figure 1 below — for 0° (0.5°) 180° that was much more detailed than that of Hipparchus. The chord lengths were based on a circle of diameter 120 units. Many of the historical details of this

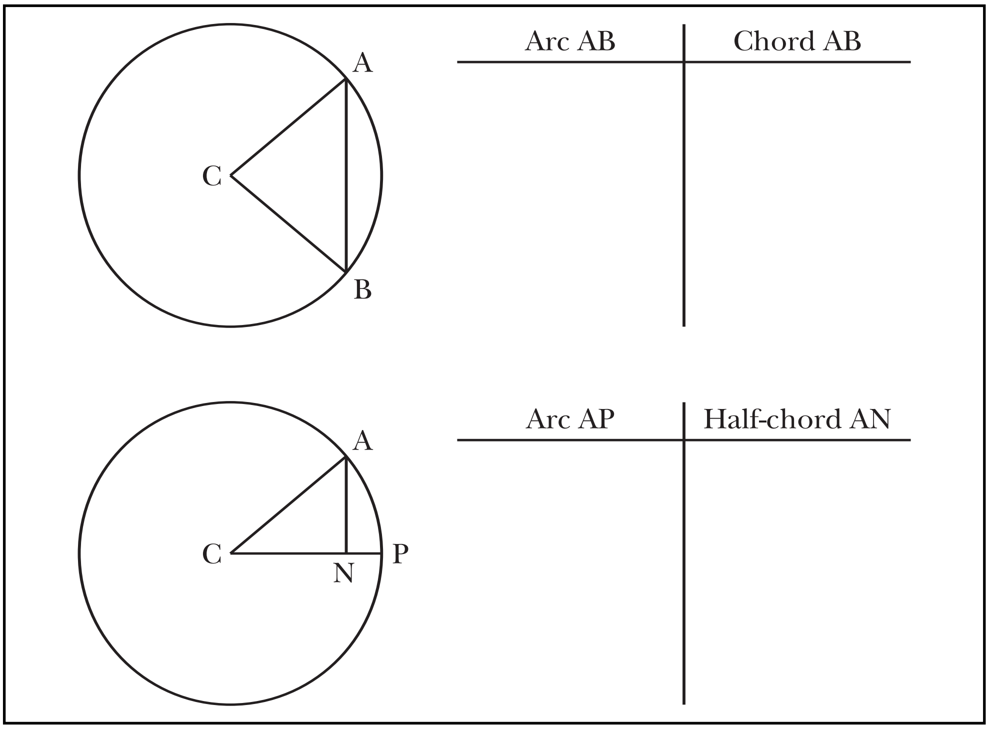

Figure 1. The layout of (a) chord and (b) half chord tables.

era are conjectural although the general trend is well established. By the end of the 16th century, as Europe drew towards the end of the 'middle ages', much more extensive tables of the trigonometric functions (chord tables) had been constructed. It is worth noting here that Napier's original logarithmic tables were based on a circle of radius 10 000 000 units. Using such radii enabled the mathematicians of those times to work in whole numbers, which they much preferred. The sine tables in the books of mathematical tables of only a few decades ago were essentially half chord tables as in lower Figure 1a above, using a circle of radius 1 unit.

In classical Greek times, the motivation for Hipparchus to construct his chord tables had been to aid him in his work in astronomy. By the 16th century, besides astronomy, we can include mechanics, surveying and cartography among other disciplines that would have benefited. Over this time, geometry and particularly trigonometry had developed. The elementary trigonometric identities were known (not necessarily in today's terminology). The identities in which we are particularly interested are:

$$2\sin\alpha\cos\beta = \sin(\alpha - \beta) + \sin(\alpha + \beta) \qquad \ldots \qquad (1)$$
$$2\cos\alpha\cos\beta = \cos(\alpha - \beta) + \cos(\alpha + \beta) \qquad \ldots \qquad (2)$$
$$2\cos\alpha\sin\beta = \sin(\alpha + \beta) - \sin(\alpha - \beta) \qquad \ldots \qquad (3)$$
$$2\sin\alpha\sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta) \qquad \ldots \qquad (4)$$

These are known as 'The Formulae of Werner', after Johannes Werner (1468–1522), a contemporary of Albrecht Dürer.

We note that in each of these formulae the left sides are all products while

the right sides are sums or differences. They can therefore be used to convert multiplication into addition or subtraction.

The above gives only a sketchy outline of the historical development but should be sufficient for our purposes whilst leaving out many fascinating facets of the story.

Applications

We shall now show how we can find the product of two numbers using identity (2) above and a table of cosines. As with sets of school mathematical tables of fifty years ago, we shall work to four significant figures so that we may show the principle without getting bogged down with too many digits. The major observatories of the late 16th century and early 17th century used tables of up to fifteen significant figures.

Let us find the product of 4562 and 6751.

$$4562 = 0.4562 \times 10^4$$
$$6751 = 0.6751 \times 10^4$$

We can now associate 0.4562 and 0.6751 with cosines of angles.

We now find 0.4562×0.6751 and adjust the decimal point later. All the angles are stated in degrees. From tables:

$$\alpha = \arccos 0.4562 = 62.86$$
$$\beta = \arccos 0.6751 = 47.54$$

Hence $$\alpha - \beta = 15.32$$
and $$\alpha + \beta = 110.40$$

$$\cos(\alpha - \beta) = 0.9645$$
$$\cos(\alpha + \beta) = -0.3486$$

Therefore $$\cos(\alpha - \beta) + \cos(\alpha + \beta) = 0.6159$$

and the product of 4562 and 6751 is $0.6159 \times 10^8 \div 2 = 30\ 800\ 000$.

This method of multiplication had been used by the Arabs as early as 1000 AD.

For a division, we can use similar methods, but we shall have to use a table of secants for one of our readings. Let us find the result of dividing 6751 by 4562.

Put $$Q = \frac{6751}{4562} = 10 \times \frac{0.6751}{4.562}$$

Associate 0.6751 with a cosine and 4.562 with a secant. From tables:

$$\alpha = \arccos 0.6751 = 47.54$$

and $$\beta = \operatorname{arcsec} 4.562 = 77.38$$

When $$Q = \frac{\cos 47.54}{\sec 77.38} \times 10 = 10 \times \cos 47.54 \cos 77.38$$

We now have a multiplication where:

$$\begin{gathered}
\alpha = 47.54 \\
\beta = 77.38 \\
\alpha - \beta = -29.84 \\
\alpha + \beta = 124.92 \\
\cos(\alpha - \beta) = 0.8674 \\
\cos(\alpha + \beta) = -0.5724 \\
\cos(\alpha - \beta) + \cos(\alpha + \beta) = 0.2950
\end{gathered}$$

Hence $$Q = 10 \times 0.2950 \div 2 = 1.475$$

References and further reading

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